Solutions

1.3: Predicate Logic

In mathematics we often express ideas in terms of some unknown variable. For instance, consider the sentence "x is even." This is not a statement because its T/F value depends on x. Predicate logic gives us a method to extend our formal system of logic to deal with this situation.

~~**Question 1.** See Inquiry 1.7 on page 28.~~

~~Blue bricks come in singles.~~

~~White bricks come in singles or doubles.~~

Question 2. Nikola bets you \$5 that every player on his basketball team will either score a point or earn an assist in tonight's game. What must happen for you to win the bet? Express this condition in the simplest, most natural way possible, and explain your reasoning.

At least one player must fail to score ~~or~~ and fail to earn an assist.

Question 3. For each of the following statements, give a list of natural numbers that satisfies the statement. Can you find a single list that satisfies both statements?

p: There is a number in the list that is greater than every other number in the list.
q: Every number in the list is less than some other number in the list.

P: {2, 4, 6, 8}   Q: {2, 4, 6, 8, 10, ...}

~~no~~ there is no list that satisfies both.

Definition 1. A predicate is a declarative sentence whose T/F value depends on one or more variables. We use function notation to denote predicates. For example,

$$P(x) = \text{“}x \text{ is even,” and}$$
$$Q(x, y) = \text{“}x \text{ is heavier than } y\text{”}$$

are predicates. Notice that if I fix the variables, say $x = x_0$ and $y = y_0$, then $P(x_0)$ and $Q(x_0, y_0)$ are bona fide statements. Implicit in any predicate is the *domain* of values that the variable(s) can take.

Quantifiers. We have just seen one way to turn a predicate into a statement. Adding a quantifier is another way. A quantifier modifies a predicate by describing whether some or all elements of the domain satisfy the predicate.

Quantifier	Name	Meaning
$\forall$	universal quantifier	for all
$\exists$	existential quantifier	there exists

Example 1. Using all cars as a domain, let

$$P(x) = \text{“}x \text{ gets good mileage,” and}$$
$$Q(x) = \text{“}x \text{ is large.”}$$

Translate the statement $(\forall x)(Q(x) \to \neg P(x))$ into English.

Every large car gets poor mileage.

Example 2. Let $P(x)$ = "x is even." Express the fact that the sum of an even number and an odd number is odd as a viable statement.

$$(\forall x)(\forall y)[(P(x) \wedge \neg P(y)) \to (\neg P(x+y))]$$

Example 3. Let $G(x, y)$ = "$x > y$." What do the following two statements mean in English? Are they the same?

(i) $(\forall y)(\exists x)G(x, y)$
(ii) $(\exists x)(\forall y)G(x, y)$

(i) No matter how large ~~y~~ a number is, there is always a larger one.

(ii) There is some number which is bigger than all other numbers.

These are related to q and p, respectively, in question 3.

Negation. Negating the quantified statement of a predicate logic is essential to proof writing.

Equivalence	**Name**
$\neg[(\forall x)P(x)] \iff (\exists x)(\neg P(x))$	universal negation
$\neg[(\exists x)P(x)] \iff (\forall x)(\neg P(x))$	existential negation

Example 4. Let $P(x)$ and $Q(x)$ be as in Example 1. Then we can rewrite the statement "all large cars get bad mileage" as $(\forall x)(Q(x) \to \neg P(x))$. What is the negation of this statement?

1. $\neg[(\forall x)(Q(x) \to \neg P(x))]$	given
2. $(\exists x)\neg(Q(x) \to \neg P(x))$	universal negation, 1
3. $(\exists x)\ \neg(\neg Q(x) \vee \neg P(x))$	implication, 2
4. $(\exists x)\ (Q(x) \wedge P(x))$	De Morgan's law and double negation

Example 5. See Example 1.14 on page 33.

$P(x) =$ "x is a pentagon."

$H(x) =$ "x is a hexagon."

$B(x,y) =$ "x borders y."

1. No two pentagons border each other.
2. Every pentagon borders a hexagon.
3. Every hexagon borders another hexagon.

1. $(\forall x)(\forall y)((P(x) \wedge P(y)) \to \neg B(x,y))$

2. $(\forall x)(P(x) \to (\exists y)(H(y) \wedge B(x,y)))$

3. $(\forall x)(H(x) \to (\exists y)(H(y) \wedge B(x,y)))$

Negate these statements for practice. It may help to think about the English first.

Two Common Constructions. There are two expressions that come up often and are worth remembering. The first is the statement

All ⟨blanks⟩ are ⟨something⟩.

$$(\forall x)(P(x) \rightarrow Q(x))$$

Negation: $(\exists x)(P(x) \wedge \neg Q(x))$

The second is the statement

There is a ⟨blank⟩ that is ⟨something⟩.

$$(\exists x)(P(x) \wedge Q(x))$$

Negation: $(\forall x)(\neg P(x) \vee \neg Q(x))$

Homework. (Due Sept 17, 2018) Section 1.3: 1, 2, 6, 10, 18

Practice Problems. Section 1.3: 3-9 (odd), 15, 16, 21, 22 **23-25**